Identifiability and Identification of Linear Time-Invariant Systems

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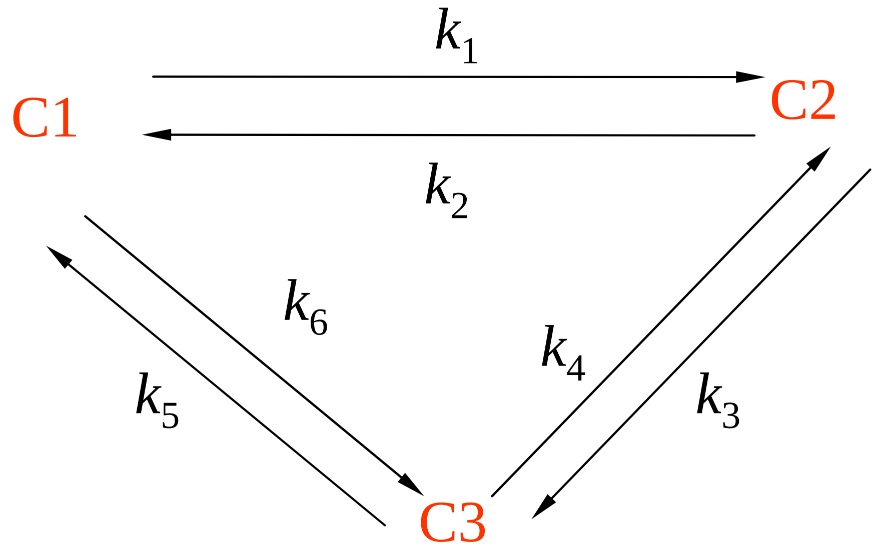
Outline

(a) Identification from initial condition responses

(b) Identification from forced responses

Example



Are the initial condition responses sufficient for uniquely determining all the chemical reaction rates?

$$\frac{dx_1}{dt} = -(k_1 + k_6)x_1 + k_2 x_2 + k_5 x_3$$

$$\frac{dx_2}{dt} = k_1 x_1 - (k_2 + k_3)x_2 + k_4 x_3$$

$$\frac{dx_3}{dt} = k_6 x_1 + k_3 x_2 - (k_5 + k_4)x_3$$

x_1, x_2, and x_3 are the concentrations of C1, C2, and C3, respectively.

$$\dot{\mathbf{x}} = A\mathbf{x}$$

System Matrix *A*

$$A = PDQ = \begin{bmatrix} p_{11} & p_{12} \cdots p_{1n} \\ p_{21} & p_{22} \cdots p_{2n} \\ \vdots & \vdots \ \ddots \ \vdots \\ p_{n1} & p_{n2} \cdots p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \cdots 0 \\ 0 & \lambda_2 \cdots 0 \\ \vdots & \vdots \ \ddots \ \vdots \\ 0 & 0 \cdots \lambda_n \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \cdots q_{1n} \\ q_{21} & q_{22} \cdots q_{2n} \\ \vdots & \vdots \ \ddots \ \vdots \\ q_{n1} & q_{n2} \cdots q_{nn} \end{bmatrix}$$

Previous research indicates that initial condition responses contain information for *D*. In order to uniquely determine A, need either P or Q, since $PQ=I$.

State Measurability

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \qquad \boldsymbol{x}(0) = [x_{10}, x_{20}, \cdots, x_{n0}]^T$$

$\boldsymbol{x}$: State Variable

NMS System: All the *n* State Variables are measurable.

OMS System: One State Variable is Measurable.

OMS System: One experiment is not enough to determine model parameters.

$\boldsymbol{V}$

$$x_i = \left[\begin{pmatrix} p_{i1}q_{11} & p_{i1}q_{12} & \cdots & p_{i1}q_{1n} \\ p_{i2}q_{21} & p_{i2}q_{22} & \cdots & p_{i2}q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{in}q_{n1} & p_{in}q_{n2} & \cdots & p_{in}q_{nn} \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{pmatrix} \right]^T \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \left[\varphi_1, \varphi_2, \cdots, \varphi_n \right] \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

$$V \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}$$

The row vectors of ***V*** have the same directions as the row vectors of ***Q***; therefore, ***V*** itself is a right eigenvector matrix of ***A***.

Example: three systems produce the same x_2 from initial concentrations [80, 300].

$$\begin{pmatrix} 80 \\ 300 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_2(t) = 126.6667e^{-t} + 173.3333e^{-4t}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -6.544837 & 0.879548 \\ -16.043139 & 1.544837 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -2.118829 & 3.796435 \\ 0.554391 & -2.881171 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

OMS System: *n* independent experiments can uniquely determine model parameters.

$$V = \begin{bmatrix} \varphi_1^1 & \varphi_1^2 \cdots \varphi_1^n \\ \varphi_2^1 & \varphi_2^2 \cdots \varphi_2^n \\ \vdots & \vdots \quad \ddots \quad \vdots \\ \varphi_n^1 & \varphi_n^2 \cdots \varphi_n^n \end{bmatrix} \begin{bmatrix} X_{10}^1 & X_{10}^2 \cdots X_{10}^n \\ X_{20}^1 & X_{20}^2 \cdots X_{20}^n \\ \vdots & \vdots \quad \ddots \quad \vdots \\ X_{n0}^1 & X_{n0}^2 \cdots X_{n0}^n \end{bmatrix}^{-1}$$

$A=V^{-1}DV$

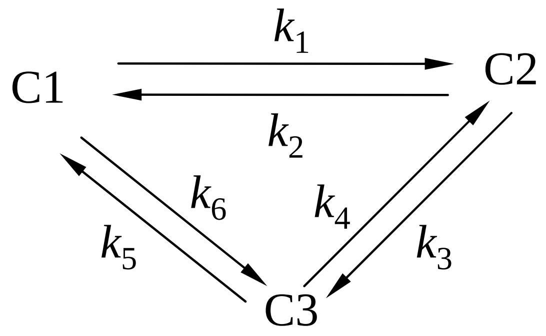
NMS System: One experiment is enough to uniquely determine model parameters.

$$\boldsymbol{x} = \begin{bmatrix} p_{11}\sum_{i=1}^{n} q_{1i}x_{i0} & p_{12}\sum_{i=1}^{n} q_{2i}x_{i0} \cdots p_{1n}\sum_{i=1}^{n} q_{ni}x_{i0} \\ p_{21}\sum_{i=1}^{n} q_{1i}x_{i0} & p_{22}\sum_{i=1}^{n} q_{2i}x_{i0} \cdots p_{2n}\sum_{i=1}^{n} q_{ni}x_{i0} \\ \vdots & \vdots \qquad \ddots \qquad \vdots \\ p_{n1}\sum_{i=1}^{n} q_{1i}x_{i0} & p_{n2}\sum_{i=1}^{n} q_{2i}x_{i0} \cdots p_{nn}\sum_{i=1}^{n} q_{ni}x_{i0} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = U \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

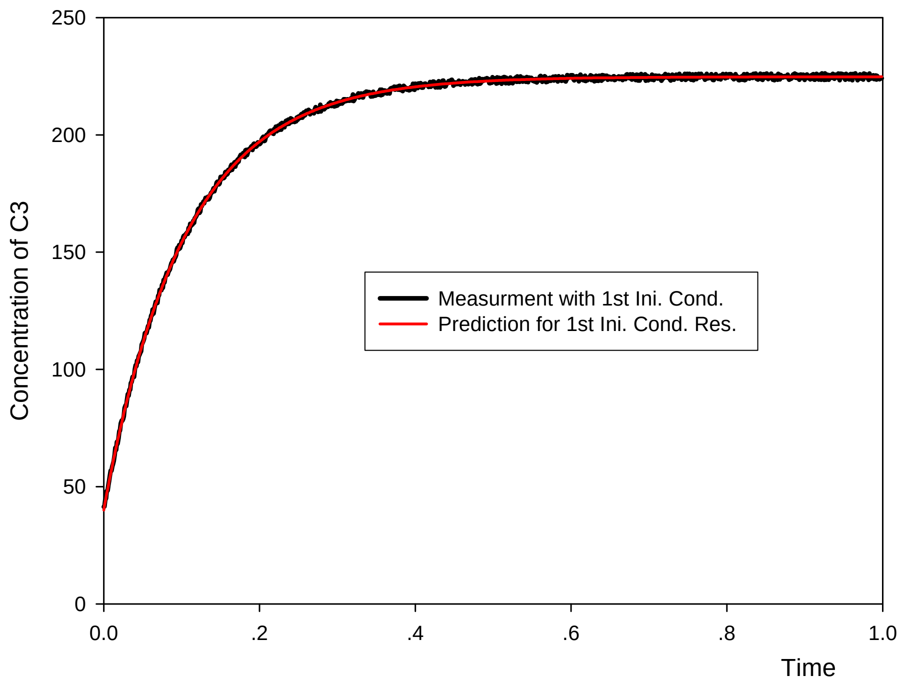
$A=UDU^{-1}$

Application of an OMS System

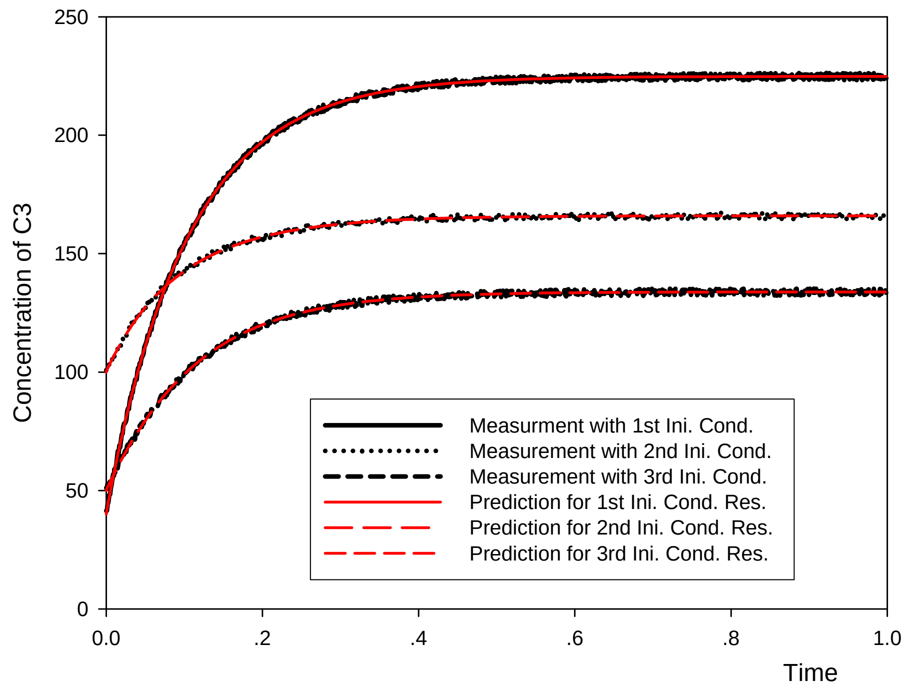
$$\mathbf{k} = [4.62, 10.34, 3.37, 5.62, 3.72, 1]$$



$$X(0) = \begin{bmatrix} 80 & 300 & 40 \\ 10 & 200 & 100 \\ 100 & 100 & 50 \end{bmatrix}$$



[4.6479, 18.0561, -2.5116, 6.7778, -1.5964, 4.8664]



[4.8021, 11.8053, 3.6070, 5.6407, 3.6240, 0.7934]

Identification from Forced Responses

System Impulse Response

$$H = e^{At}B = \begin{bmatrix} \sum_{i=1}^{n} p_{1i} \sum_{j=1}^{n} q_{ij} b_{j1} e^{\lambda_i t} & \sum_{i=1}^{n} p_{1i} \sum_{j=1}^{n} q_{ij} b_{j2} e^{\lambda_i t} & \cdots & \sum_{i=1}^{n} p_{1i} \sum_{j=1}^{n} q_{ij} b_{jp} e^{\lambda_i t} \\ \sum_{i=1}^{n} p_{2i} \sum_{j=1}^{n} q_{ij} b_{j1} e^{\lambda_i t} & \sum_{i=1}^{n} p_{2i} \sum_{j=1}^{n} q_{ij} b_{j2} e^{\lambda_i t} & \cdots & \sum_{i=1}^{n} p_{2i} \sum_{j=1}^{n} q_{ij} b_{jp} e^{\lambda_i t} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} p_{ni} \sum_{j=1}^{n} q_{ij} b_{j1} e^{\lambda_i t} & \sum_{i=1}^{n} p_{ni} \sum_{j=1}^{n} q_{ij} b_{j2} e^{\lambda_i t} & \cdots & \sum_{i=1}^{n} p_{ni} \sum_{j=1}^{n} q_{ij} b_{jp} e^{\lambda_i t} \end{bmatrix}$$

The polynomial structure of the H cells reveals

(1) A left eigenvector matrix of A can be directly determined from the s^{th} column of H, implying structural identifiable if all state variables are measurable.

(2) One row vector of H allows unique determination of a right eigenvector matrix for A if B has n linearly independent known column vectors, indicating structural identifiable if there is one measurable state variable when B is full rank.

Identifiability from Forced and Initial Condition Responses

Both external excitation and initial condition affect system dynamic responses. A logical question is whether initial condition response and forced response provide independent constraints for model parameter determination.

$$\mathrm{HX}_{\mathrm{X}^0} = \left[e^{\mathrm{A}t}\mathrm{B} \quad e^{\mathrm{A}t}\mathrm{X}^0 \right] = e^{\mathrm{A}t}\overline{\mathrm{B}}$$

OMS System Identification from Forced and Initial Condition Responses

$$\begin{bmatrix} hx_{X0}^{r1} \\ hx_{X0}^{r2} \\ \vdots \\ hx_{X0}^{r(p+l)} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} p_{ri} \sum_{j=1}^{n} q_{ij} \bar{b}_{j1} e^{\lambda_i t} \\ \sum_{i=1}^{n} p_{ri} \sum_{j=1}^{n} q_{ij} \bar{b}_{j2} e^{\lambda_i t} \\ \vdots \\ \sum_{i=1}^{n} p_{ri} \sum_{j=1}^{n} q_{ij} \bar{b}_{j(p+l)} e^{\lambda_i t} \end{bmatrix} = \left[\begin{pmatrix} p_{r1}q_{11} & p_{r1}q_{12} \cdots p_{r1}q_{1n} \\ p_{r2}q_{21} & p_{r2}q_{22} \cdots p_{r2}q_{2n} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ p_{rn}q_{n1} & p_{rn}q_{n2} \cdots p_{rn}q_{nn} \end{pmatrix} \overline{B} \right]^T \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \varphi \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

$$\mathrm{V} = \varphi^T \overline{\mathrm{B}}^T (\overline{\mathrm{B}}\,\overline{\mathrm{B}}^T)^{-1}$$

$$\mathrm{A} = \mathrm{V}^{-1} \mathrm{D} \mathrm{V}$$

Conclusions

(1) If only one concentration is measurable, initial condition response from one experiment can not uniquely determine system parameters. n experiments with independent initial conditions are required.

(2) If all the chemical concentrations are measurable, initial condition response from one experiment is enough to uniquely determine system parameters.

Conclusions

(3) If the control matrix has *n* known independent column vectors or if all the state variables are measurable, an LTI system is identifiable from forced responses.

(4) Initial condition response can provide complementary and independent constraints to those from forced responses. Combining forced and initial condition responses can improve parameter identifiability.